Hopefully Helpful Hints for Gauss's Law

As before, there are things you need to know about Gauss's Law. In no particular order, they are:

a.) In the context of Gauss's Law, at a differential level, the electric flux is related to the amount of electric field E that passes through a differential surface dA. It is defined as:

$$d\Phi_e = \vec{E} \bullet d\vec{A},$$

where $d\vec{A}$ is a contrived vector whose direction is perpendicular to and away from the surface (that is, it's at right angles to the face of the surface) and whose magnitude it equal to the differential surface area of the surface.

b.) What Gauss observed was that if there is a *closed* surface, the amount of electric flux *through* the surface will be proportional to the amount of charge *inside* the surface.

1.)

In other words, if you sum up all the differential fluxes through all of the differential surfaces over the entire closed structure, you will find that that number you end up with will be proportional to the amount of charge *inside* the structure. Mathematically, this is stated as:

$$\Phi_e \ \alpha \ q_{enclosed}$$

In expanded form, this comes out as:

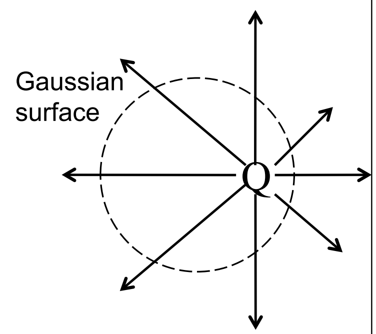
$$\int_S \vec{E} \bullet d\vec{A} \ \alpha \ q_{enclosed}$$

To make this into an equality, you have to multiply by a proportionality constant. In this case, it is the *permittivity of free space*, or ε_o. With that, Gauss's Law is written as:

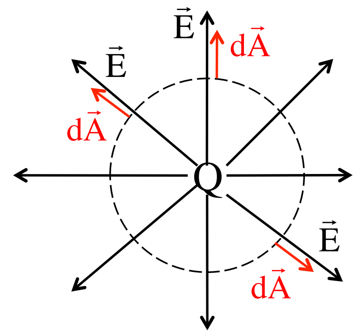
$$\int_S \vec{E} \bullet d\vec{A} = \frac{q_{enclosed}}{\varepsilon_o}$$

2.)

b.) Gauss's Law is ALWAYS TRUE, whether the integral is easy to do or not (a charge sitting next to the edge of a sphere will satisfy the law, but with the electric field due to the charge varying from point to point on the surface, and with the angle between the electric field vector and the area vector varying, God help the poor soul who tries to evaluate the integral).



The trick to using Gauss's Law is to find a geometry that is symmetric with the charge. That is, you want a geometry that either has the electric field vectors parallel to the area vectors, or one in which the dot product between the two parameters is zero (that is, the two are at right angles with one another—this will be important when we get to cylindrical symmetry).



c.) As an interesting side note: Although the integral on the left side of Gauss's Law looks like the nasty part of a problem (note the nastiness),

$$\oint_S \vec{E} \bullet d\vec{A} = \frac{q_{enclosed}}{\varepsilon_o}$$

for spherical symmetry that integral will ALWAYS be of the same form (and the same is true of cylindrical symmetry). In other words, when you start doing more complicated problems, the hard part is determine how much charge is inside the imaginary Gaussian surface.

d.) Just to be complete, the left side will always look like:

$E\left(4\pi r^2\right)$ for spherical symmetry, where "r" is the Gaussian sphere's radius

$E\left(2\pi rL\right)$ for cylindrical symmetry, where "r" is the Gaussian cylinder's radius

NOTE: For both the AP test and the Gauss's Law test, you should be able to derive both of these relationships from scratch!

2.) As was said above, the difficulty of most Gauss's Law problems is determining the "charge enclosed" part of the relationship. You have to include ALL the charge involved, and in some cases you have to do some pretty fancy dancing to execute that feat. The following are examples of problems with spherical symmetry and cylindrical symmetry.

3.) Problem with SPHERICAL SYMMETRY: Assume your have a thick skinned, insulating (non-conducting), hollow sphere of inside radius *a* and outside radius *b* (a cross-section is shown on the next page). Assume also that the insulator has a volume-charge-density in it that is non-linear and is defined as:

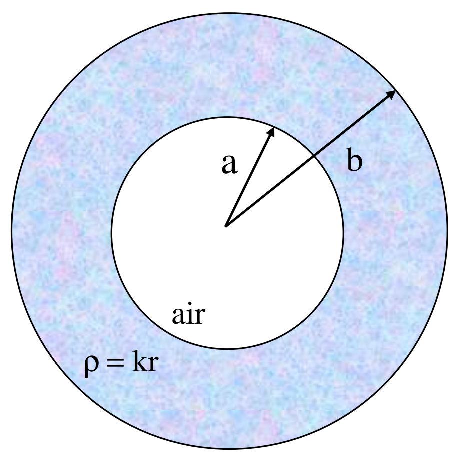
$$\rho = k_1 r, \text{ where "}k_1\text{" is a constant (not } \frac{1}{4\pi\varepsilon_o}).$$

Determine:

a.) $E(r)$ for $r < a$;

b.) $E(r)$ for $a < r < b$;

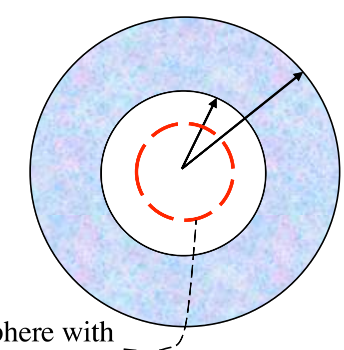
c.) $E(r)$ for $r > b$;

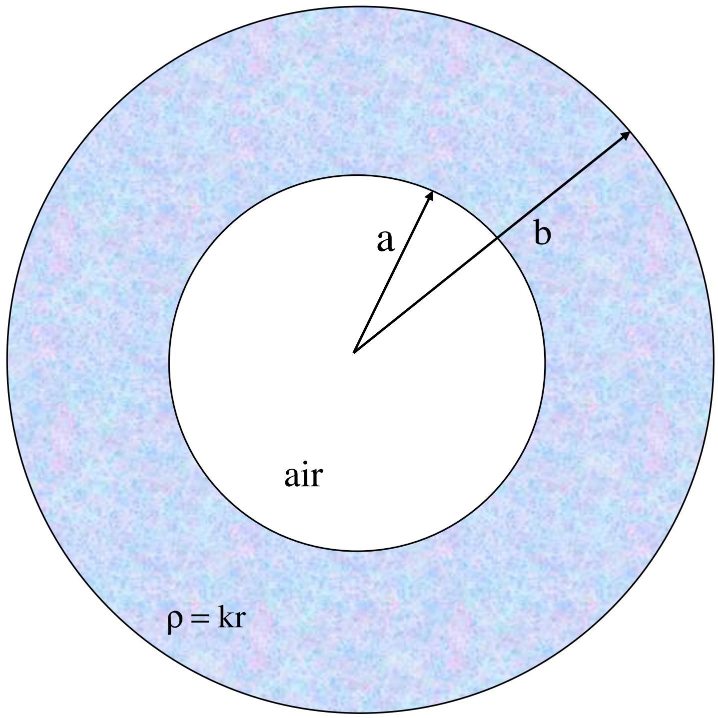


(Note that this is, quite literally, how most Gauss's Law problems are stated on AP tests.)

a.) $E(r)$ for $r < a$;

If we place an imaginary, spherical, Gaussian surface of radius less than "a" centered at the configuration's center, the charge enclosed within the Gaussian surface will be zero. That mean the electric field in that region will be zero inside that region.





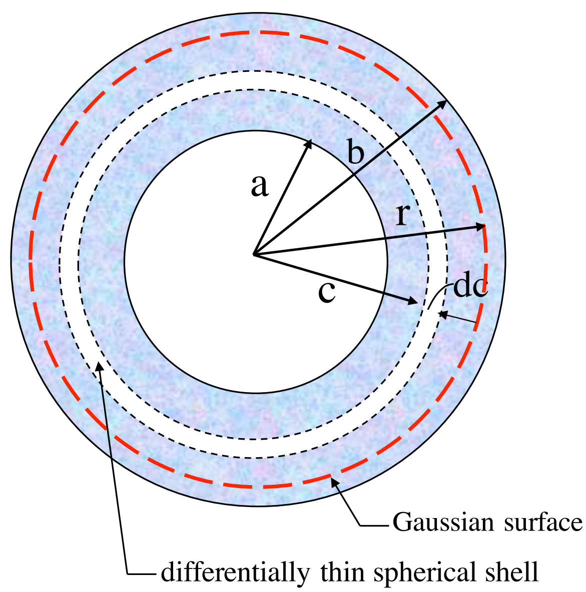
7.)

b.) $E(r)$ for $a < r < b$;

Define a Gaussian surface between *a* and *b*. Call it's radius *c* and its differential thickness "dc."

Being a sphere, the shell's surface area will be $4\pi c^2$.

Again, being a sphere, the shell's differential volume will equal the product of the shell's surface area and thickness (*dc*). That is:

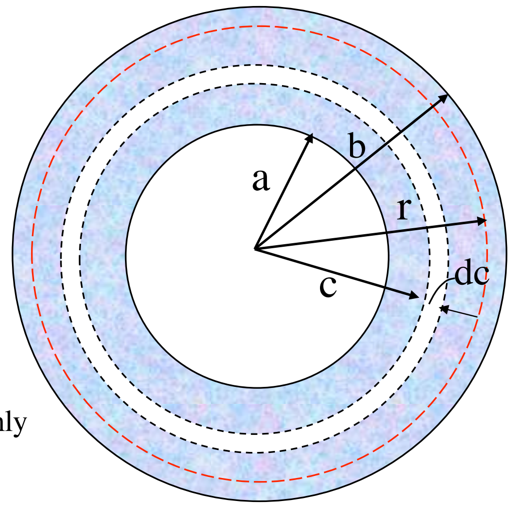


$$dV = (\text{surface area})(\text{differential thickness})$$
$$= \quad (4\pi c^2) \qquad\qquad (dc)$$

8.)

The differential charge dq inside will be:

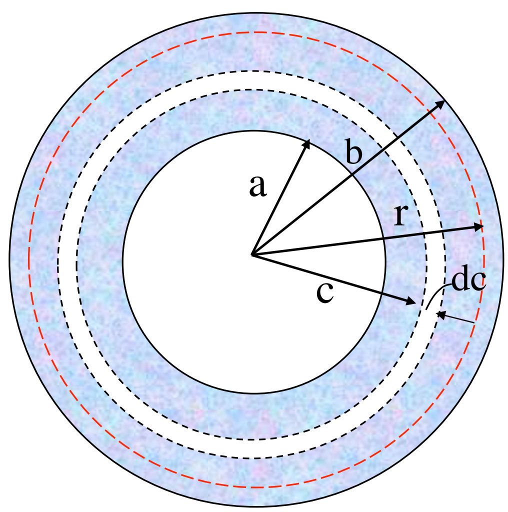
$$\begin{aligned} dq &= (\text{volume charge density evaluated at "c"})(\text{differential volume}) \\ &= \quad (\rho) \quad (4\pi c^2\ dc) \\ &= \quad (kc) \quad (4\pi c^2\ dc) \\ &= (4\pi k)(c^3\ dc) \end{aligned}$$



differential charge dq uniformly distributed throughout shell

Knowing the differential charge

$$\begin{aligned} dq &= (k_1 c)(4\pi c^2\ dc) \\ &= 4\pi k_1 c^3\ dc, \end{aligned}$$



we can sum up all the differential charge amounts inside all the differential volumes between *a* and *r* (that is, everywhere there is charge inside the Gaussian surface) and write Gauss's Law for *E* between *a* and *r* as:

$$\int_{c=a}^{r} E \bullet dA = \frac{q_{encl}}{\varepsilon_o}$$

This is evaluated fully on the next page.

$$\oint_S \vec{E} \bullet d\vec{A} = \frac{q_{encl}}{\varepsilon_o}$$

$$\Rightarrow \quad \oint_S |\vec{E}||d\vec{A}|\cos 0^\circ = \frac{\int dq}{\varepsilon_o}$$

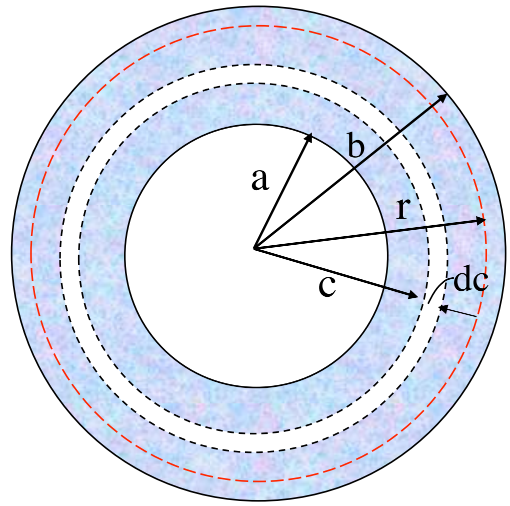
$$|\vec{E}|\oint_S |d\vec{A}| = \frac{\int_{c=a}^{r} 4\pi k_1 c^3\, dc}{\varepsilon_o}$$

$$|\vec{E}|\left(4\pi r^2\right) = \frac{4\pi k_1}{\varepsilon_o}\int_{c=a}^{r} c^3\, dc$$

$$\Rightarrow \quad |\vec{E}| = \left(\frac{1}{4\pi r^2}\right)\frac{4\pi k_1}{\varepsilon_o}\left[\frac{c^4}{4}\right]_{c=a}^{c=r}$$

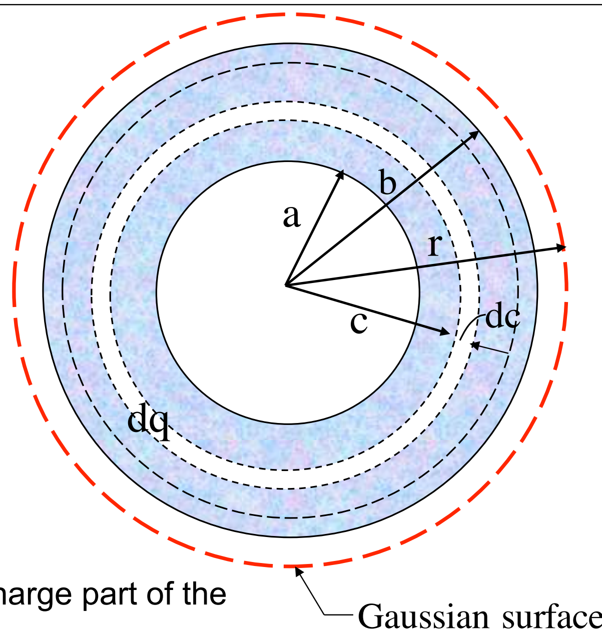
$$\Rightarrow \quad |\vec{E}| = \frac{k_1}{\varepsilon_o r^2}\left[\frac{r^4}{4} - \frac{a^4}{4}\right]$$

$$\Rightarrow \quad |\vec{E}| = \frac{k_1}{4\varepsilon_o r^2}\left[r^4 - a^4\right]$$



c.) $E(r)$ for $r > b$;

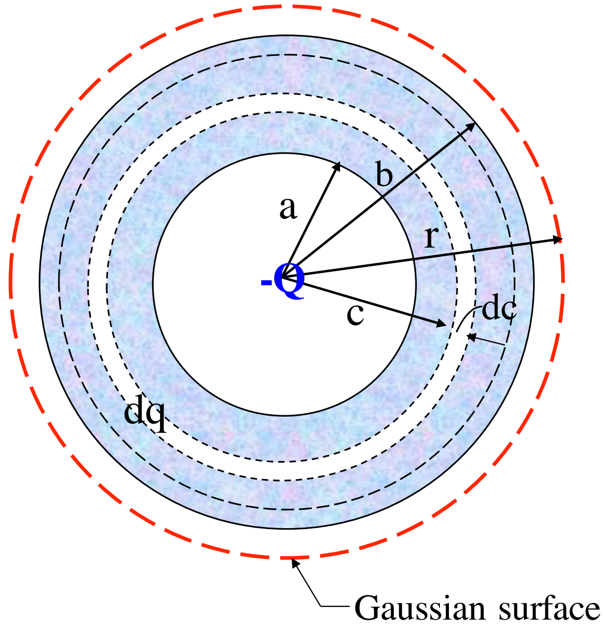
Now the Gaussian radius *r* is greater than *b*. The ONLY difference between this and the previous section is that the charge integral will go from *c=a* to *c=b* instead of from *c=a* to *c=r* (this is because *r* is now *outside* the sphere, so the charge enclosed is ALL of the charge inside the sphere). By simply putting a *b* wherever we see an *r* in the charge part of the equation, we get:



$$|\vec{E}| = \frac{k_1}{4\varepsilon_o r^2}\left[b^4 - a^4\right]$$

Note: The *r* in the denominator is from the flux part (it's the Gaussian radius), not the "charge enclosed" part, so it doesn't change!

ADDED TWIST: Let's assume we still wanted the field outside of *b*, but in addition to the volume charge density in the insulator section there is a negative charge -Q at the center? How would things have changed?



The only difference would have been on the right side of Gauss's Law in the "charged enclosed" term. That would have been:

$$= \frac{-Q + \int dq}{\varepsilon_o}$$

The math for this situation is shown on the next page. What important thing to observe here is that ALL the charge inside the Gaussian surface that has be accounted for!!!

13.)

So why do I always assume the electric field to be in the direction of *dA*?

In the solution, notice that there is a positive and negative part (look at it!). When you start the problem, YOU DON'T KNOW which one will be larger. As a consequence, you won't know whether the electric field through the Gaussian surface will be inward or outward.

In cases like this, you have to chose one or the other. For simplicity, I ALWAYS choose *outward* along the line of *dA*. IF I'M WRONG, all that happens is that when I put numbers in, I get a negative sign in front of the electric field magnitude. Because magnitudes shouldn't be negative, that sign lets me know I've assumed the wrong direction for *E*.

$$\oint_S \vec{E} \bullet d\vec{A} = \frac{-Q + \int dq}{\varepsilon_o}$$

$$= \frac{-Q + \int_{c=a}^{b} 4\pi k_1 c^3\, dc}{\varepsilon_o}$$

$$= \frac{-Q}{\varepsilon_o} + \frac{4\pi k_1}{\varepsilon_o} \int_{c=a}^{b} c^3\, dc$$

$$= \frac{-Q}{\varepsilon_o} + \frac{4\pi k_1}{\varepsilon_o} \frac{c^4}{4} \Big|_{c=a}^{c=b}$$

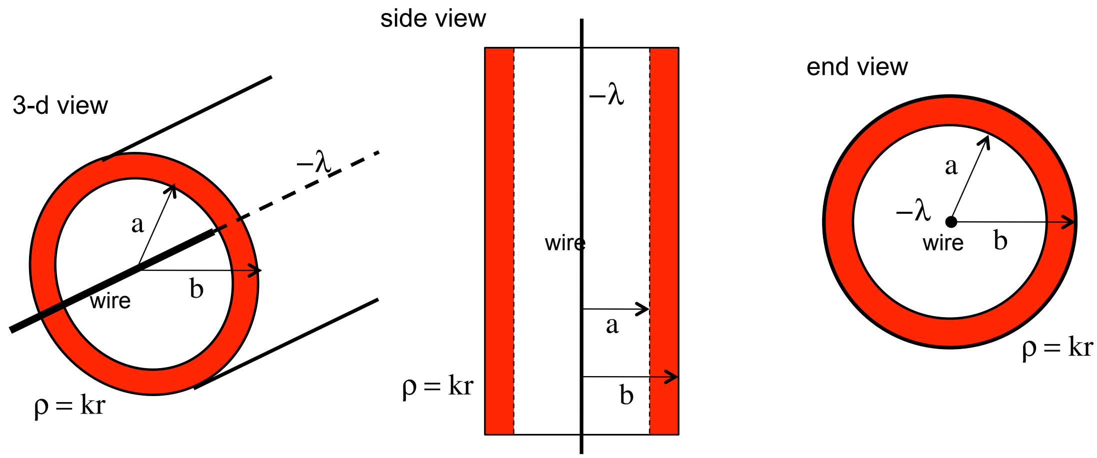
$$= \frac{-Q}{\varepsilon_o} + \frac{4\pi k_1}{\varepsilon_o} \left[\frac{b^4}{4} - \frac{a^4}{4} \right]$$

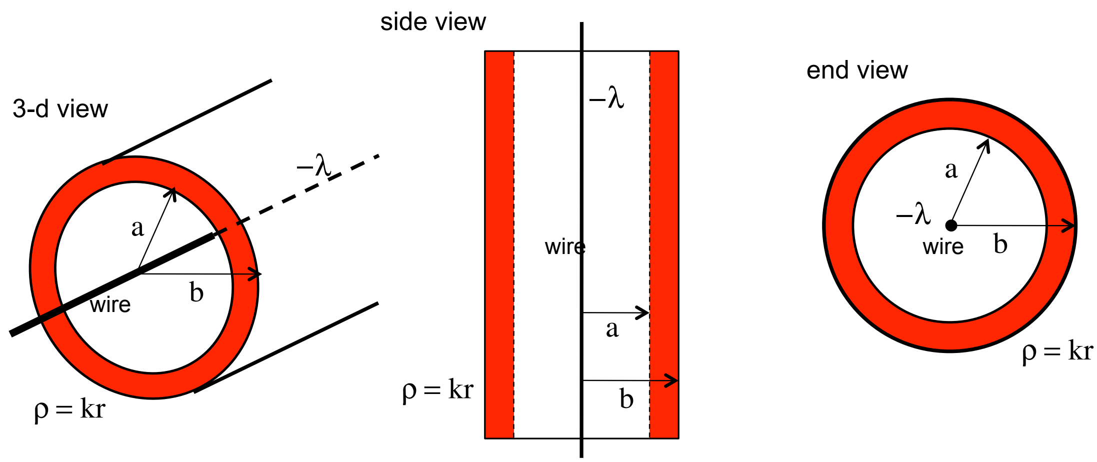
$$= \frac{-Q}{\varepsilon_o} + \frac{\pi k_1}{\varepsilon_o} \left[b^4 - a^4 \right]$$

$$E\left(4\pi r^2\right) = \frac{-Q}{\varepsilon_o} + \frac{\pi k_1}{\varepsilon_o} \left[b^4 - a^4 \right]$$

$$\Rightarrow \quad E = \frac{-Q}{4\pi\varepsilon_o r^2} + \frac{k_1}{4\varepsilon_o r^2} \left[b^4 - a^4 \right]$$

4.) CYLINDRICAL SYMMETRY: Consider now a thick-skinned insulating cylinder with inside radius *a* and outside radius *b*. Assume there is $-\lambda$'s worth of charge on a very thin wire down its axis, and in the insulating material a volume charge density of $\rho = k_1 r$, where "k_1" is a positive constant. Three different views of our system are shown below.





Our problem is to derive an expression for:

a.) $E(r)$ for $r < a$;

b.) $E(r)$ for $a < r < b$;

c.) $E(r)$ for $r > b$;

a.) $E(r)$ for $r < a$: As always, we begin by generating an imaginary Gaussian surface that has one of two properties. Either:

a.) the magnitude of the electric field is the same at every point on the surface with a directional vector relationship with *dA* that doesn't vary, or;

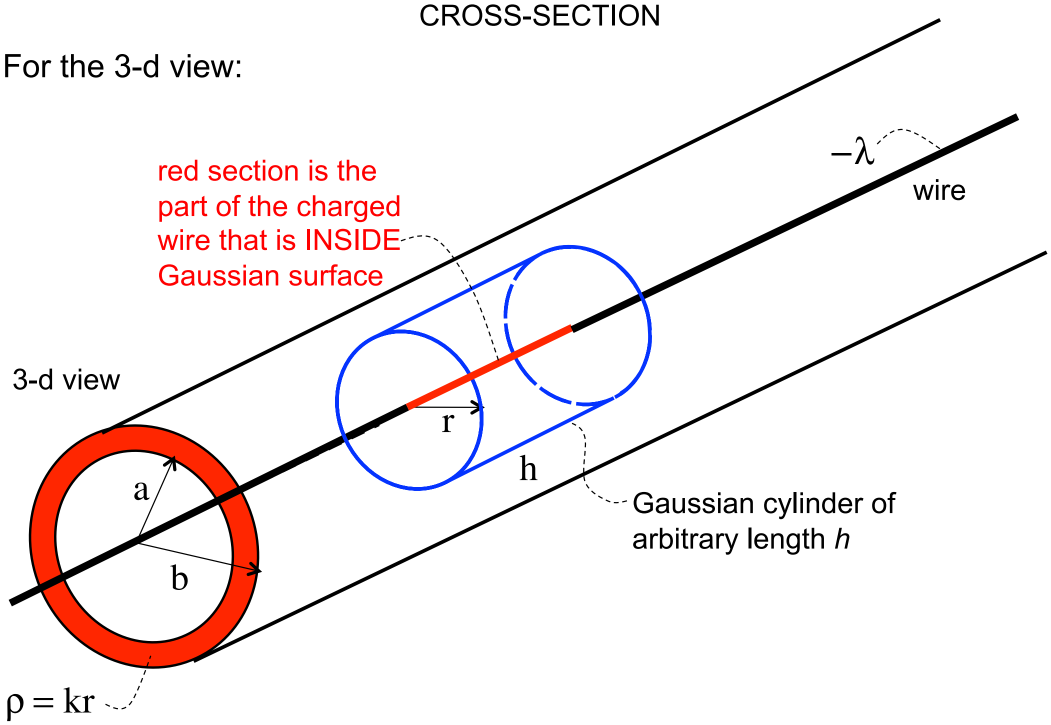
b.) the dot product of *E* and *dA* evaluated on the surface is zero (that is, *E* runs along the face, not perpendicular to the face).

For cylindrical symmetry, the surface that does the trick is . . . wait for it . . . a *cylinder*! That Gaussian surface shown on the next three pages from all three views.

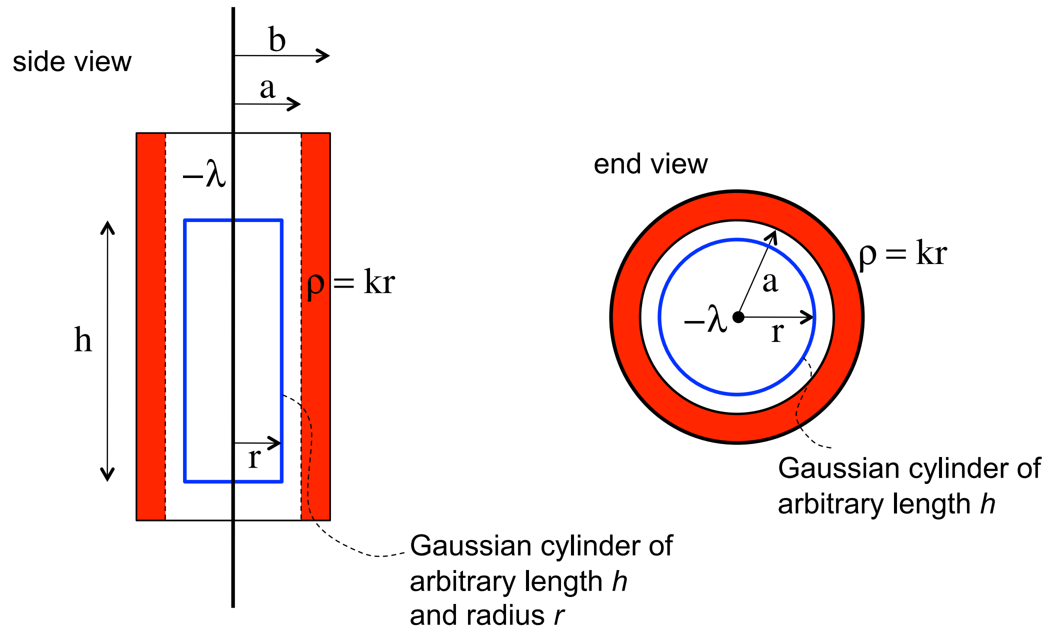
(On the next several pages, that Gaussian surface will be shown from all three perspectives.)

CROSS-SECTION

For the 3-d view:



CROSS-SECTIONS



The amount of charge found inside the Gaussian surface is equal to the charge-per-unit-length $(-\lambda)$ times the length *h* of the Gaussian surface (that is, $q_{enclosed} = -\lambda h$). Noting that the cylindrical part of the cylinder has flux through it but the end-caps don't (the angle between *E* and *dA* at the end-caps is ninety degrees), the sum of all the differential *dA*'s over the cylindrical part of the Gaussian surface will be equal to the circumference of that Gaussian cylinder $(2\pi r)$ times its length *h*, or $(2\pi r)h$. Assuming (as always) that *dA* and *E* are in the same direction (even though we know they aren't in this case as the field-producing charge is negative—remember, we are going for consistency here with our direction assumptions), we can write Gauss's Law as shown to the right:

$$\int \vec{E} \bullet d\vec{A} = \frac{q_{enclosed}}{\varepsilon_o}$$

$$\Rightarrow \int \left|\vec{E}\right|\left|d\vec{A}\right|\cos\theta = \frac{(-\lambda h)}{\varepsilon_o}$$

$$\Rightarrow \left|\vec{E}\right|\int\left|d\vec{A}\right| = \frac{(-\lambda h)}{\varepsilon_o}$$

$$\Rightarrow \left|\vec{E}\right|(2\pi r\ h) = \frac{(-\lambda h)}{\varepsilon_o}$$

$$\Rightarrow \left|\vec{E}\right| = -\frac{\lambda}{2\pi\varepsilon_o r}$$

Remember, a calculated –E means only that the assumed direction for *E* was wrong —no big deal! In fact, we expected that for this relatively simple configuration.)

b.) determine $E(r)$ for $a < r < b$:

The imaginary, cylindrical, Gaussian surface now must extend into the insulating material. That means we are going to have two charge sources within the Gaussian surface. Getting the insulator's charge (inside the Gaussian surface) is a little tricky. Follow along:

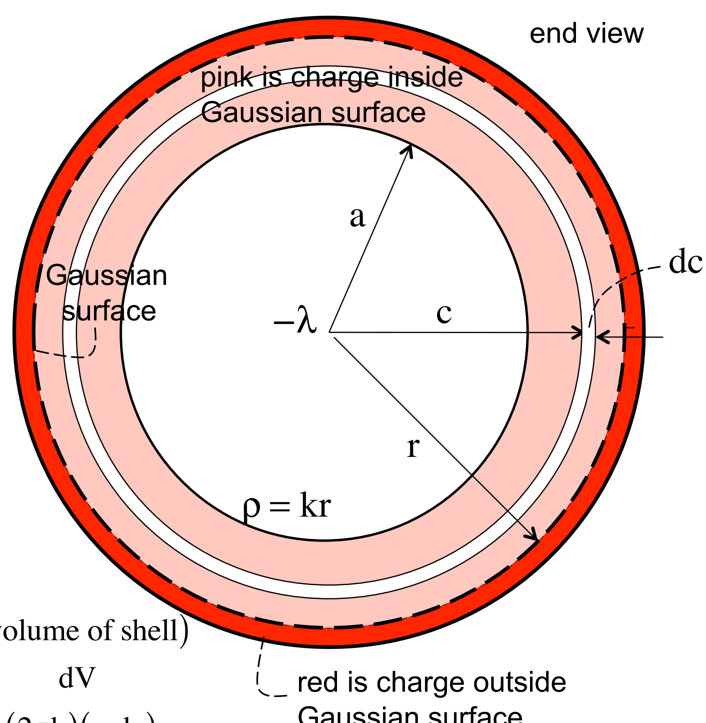
We begin by defining a differentially thin cylindrical shell of radius *c* and thickness dc located inside both the insulator and inside the Gaussian radius *r*.

If we can determine how much differential charge *dq* is in that differentially thin shell, we can integrate over all the shells from *c=a* (where the insulating material starts) to *c=r* (where the Gaussian surface is) to get the insulator part of the charge inside the Gaussian surface.

The end-view sketch on the next page highlights all of this.

The differential volume *dV* of a cylindrical shell of length *h*, radius *c* and thickness *dc* is equal to the circumference times the thickness times the length, or:

$$
\begin{aligned}
dV &= (\text{circ.})(\text{length})(\text{thickness}) \\
&= (2\pi c) \quad (h) \quad (dc) \\
&= (2\pi h)(c\ dc)
\end{aligned}
$$



The differential charge *dq* in that differentially thin cylindrical shell (again, thickness *dc*) is equal to:

$$
\begin{aligned}
dq &= (\text{charge per unit volume})(\text{volume of shell}) \\
&= \rho \qquad\qquad dV \\
&= (k_1 c) \qquad (2\pi h)(c\ dc) \\
&= (2\pi k_1 h)(c^2 dc)
\end{aligned}
$$

The total charge inside the Gaussian surface will be the charge on the wire (i.e., $-\lambda h$) added to the net charge inside the insulating, differential, cylindrical shells between *a* and *r*. In other words, it will be the evaluation of the integral $\int \rho dV$. With all this, we can write Gauss's Law as:

$$\int \vec{E} \bullet d\vec{A} = \frac{q_{enclosed}}{\varepsilon_o}$$

$$\Rightarrow \quad \int |\vec{E}||d\vec{A}| \cos\theta = \frac{(-\lambda h) + \int \rho dV}{\varepsilon_o}$$

$$\Rightarrow \quad |\vec{E}| \int |d\vec{A}| = \frac{(-\lambda h) + \int_{c=a}^{r} (k_1 c)(2\pi h\ c\ dc)}{\varepsilon_o}$$

$$\Rightarrow \quad |\vec{E}|(2\pi r\ h) = \frac{(-\lambda h) + 2\pi h k_1 \int_{c=a}^{r} c^2\, dc}{\varepsilon_o}$$

$$\Rightarrow \quad |\vec{E}|(2\pi r\ h) = \frac{(-\lambda h) + 2\pi h k_1 \left[\frac{c^3}{3}\right]\Big|_{c=a}^{r}}{\varepsilon_o}$$

$$\Rightarrow \quad |\vec{E}|(2\pi r\ h) = \frac{(-\lambda h) + \frac{2}{3}\pi h k_1 \left[r^3 - a^3\right]}{\varepsilon_o}$$

$$\Rightarrow \quad |\vec{E}| = \frac{(-\lambda) + \frac{2}{3}\pi k_1 \left[r^3 - a^3\right]}{2\pi\varepsilon_o r}$$

c.) $E(r)$ for $r > b$: Just as was the case with the spherical symmetry problem, the only difference between the magnitude of *E* in the outside region and in the insulating region is the limits used when doing the integral that determines the "charge enclose." As before, those limits should be *c=a* to *c=b* instead of *c=a* to *c=r*. Making that alteration, you can make the appropriate adjustments to the solution from the previous section and end up with the solution given below.

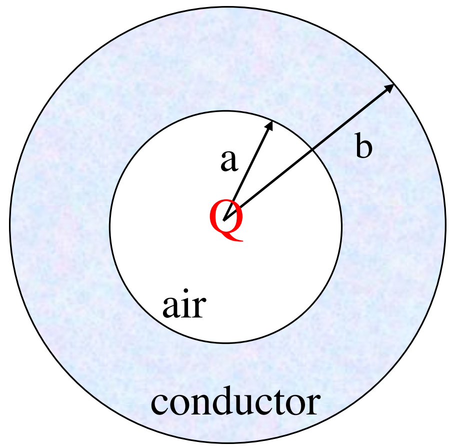
$$|\vec{E}| = \frac{(-\lambda) + \frac{2}{3}\pi k_1 \left[b^3 - a^3\right]}{2\pi\varepsilon_o r}$$

Note that the only time you really *can't* make simple adjustments like this is when you've done some canceling that is no longer applicable. For that you have to be careful.

5.) There is one more thing we need to say something about when it comes to both spherical and cylindrical symmetry. That has to do with conductors.

If you will remember, conductors in electrostatic situations cannot have electric fields within them. Otherwise, free charge will respond to those electric fields and move around until the motion-producing fields are neutralized.

So consider the following problem: You have a positive point charge Q suspended from an ignorable string and located at the center of a hollow conducting sphere of inside radius *a* and outside radius *b*. Use Gauss's Law on the region inside the conductor and see what you get.



Dutifully, we begin the Gauss's Law process by generating an imaginary Gaussian surface or radius *r* between *a* and *b* (see sketch).

With the sketch, we execute Gauss's Law:

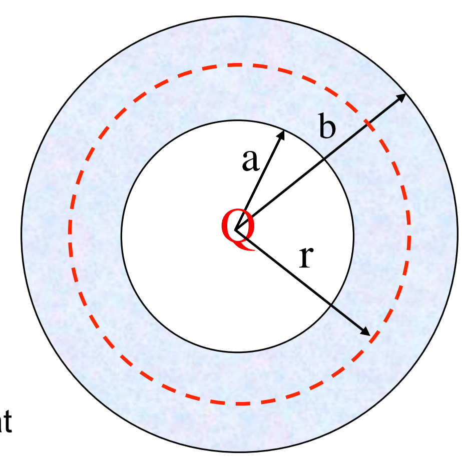
$$\int \vec{E} \bullet d\vec{A} = \frac{q_{enclosed}}{\varepsilon_o}$$

$$\Rightarrow \quad \int |\vec{E}||d\vec{A}|\cos\theta = \frac{Q}{\varepsilon_o}$$

$$\Rightarrow \quad |\vec{E}|\int |d\vec{A}| = \frac{Q}{\varepsilon_o}$$

$$\Rightarrow \quad |\vec{E}|\left(4\pi r^2\right) = \frac{Q}{\varepsilon_o}$$

$$\Rightarrow \quad |\vec{E}| = \frac{Q}{4\pi\varepsilon_o r^2}$$



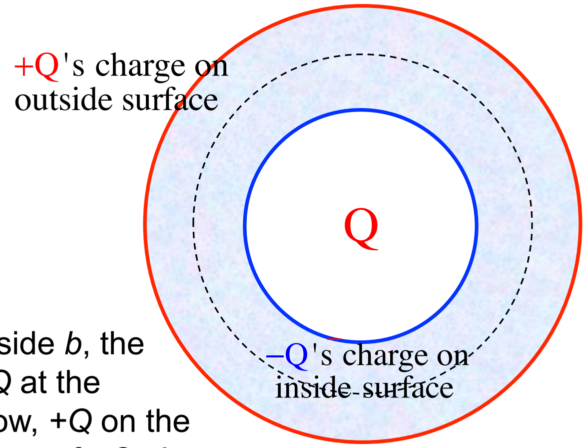
Oopsie! This suggests there's a non-zero electric field inside the conductor, but we know that can't happen. So what gives?

The key is in the fact that electrons can move around freely inside the conductor. In this case, -*Q*'s worth of charge will migrate to the *inside surface* of the hollow sphere leaving +*Q*'s worth of charge on the *outside surface*. Then, when the "charge enclosed" is determine for the situation between *a* and *b*, the net charge inside the Gaussian surface is zero and we deduce there is no electric field in the region. That is:

$$\int \vec{E} \bullet d\vec{A} = \frac{q_{enclosed}}{\varepsilon_o}$$

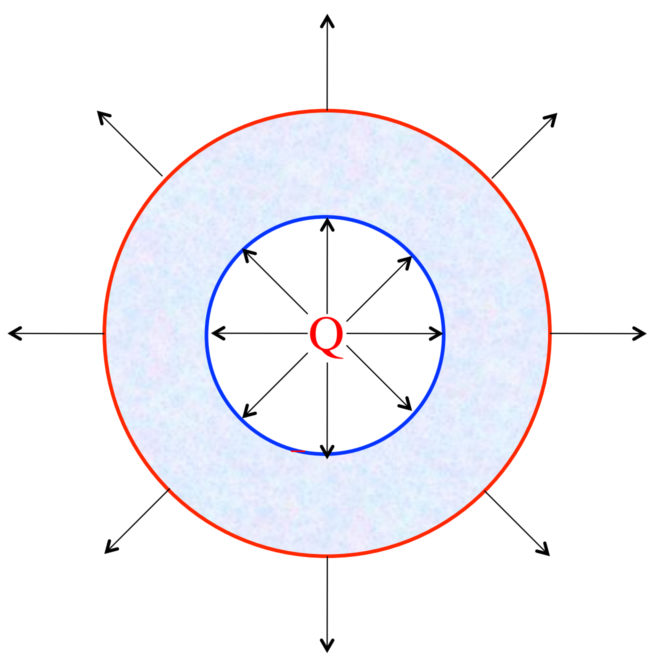
$$\Rightarrow \quad \int |\vec{E}||d\vec{A}| \cos\theta = \frac{Q + (-Q)}{\varepsilon_o} = 0$$

$$\Rightarrow \quad |\vec{E}| = 0$$



When you deal with the region outside *b*, the total "charge enclose" becomes +*Q* at the center, -*Q* on the inside of the hollow, +*Q* on the outside of the sphere for a net charge of +Q. In other words, from a distance the configuration will look like a point charge.

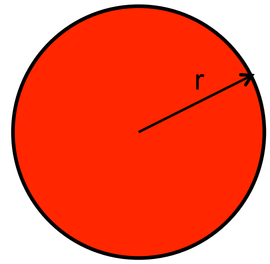
As an interesting side-note, the electric field lines for this problem would look like:



Relationships you should remember!

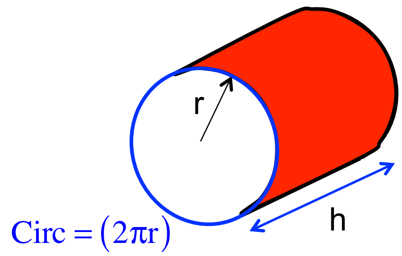
Surface area for a sphere:

$$S = 4\pi r^2$$



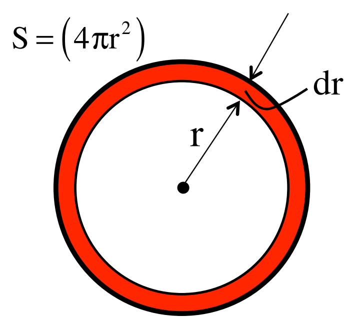
Surface area for barrel of a cylinder:
(the circumference times the length)

$$S = (2\pi r)h$$



Differential volume dV for a spherical shell:
(this is the sphere's surface area S times its thickness dr)

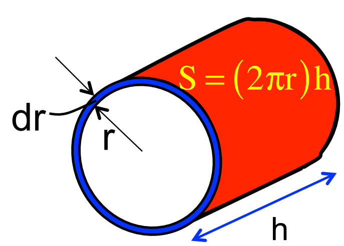
$$dV = (4\pi r^2)dr$$



cross-section of spherical shell

Differential volume dV of a differentially thin cylindrical shell: (this is the cylinder's surface area S times its thickness dr)

$$dV = \quad S \quad dr$$
$$= [(2\pi r)h] \; dr$$



Differential charge dq inside differentially thin spherical shell of volume dV:
(this is the differential volume of the shell times its charge density)

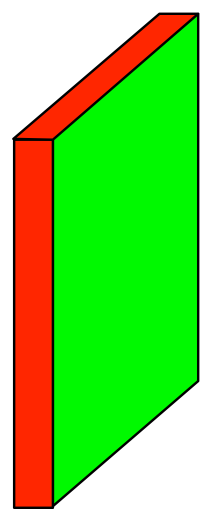
$$\begin{aligned} dq &= \rho dV \\ &= \rho\left[\left(4\pi r^2\right)dr\right] \end{aligned}$$

Differential charge dq inside differentially thin cylindrical shell of volume dV:
(this is the differential volume of the shell times its charge density)

$$\begin{aligned} dq &= \rho dV \\ &= \rho\left[\left(2\pi rh\right)dr\right] \end{aligned}$$

Lastly, what about a thin sheet of a conductor material with a charge-per-unit-area σ on it?

To begin with, when talking about a conductor, σ denotes the amount of charge per unit area on ONE SURFACE of the structure. That means there is σ's worth of charge-per-unit-area on one side of the sheet, and σ's worth of charge-per-unit-area on the other side of the sheet.



Remember, Gaussian surfaces are chosen so as to have one of two general characteristics. Either:

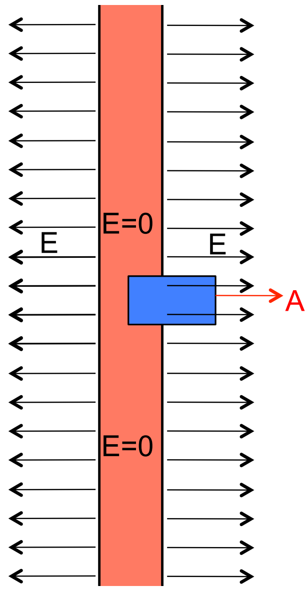
1.) The magnitude of E is the same everywhere on the surface, and E and the differential area vector dA have the same angle between them everywhere on the surface (preferably zero degrees), OR

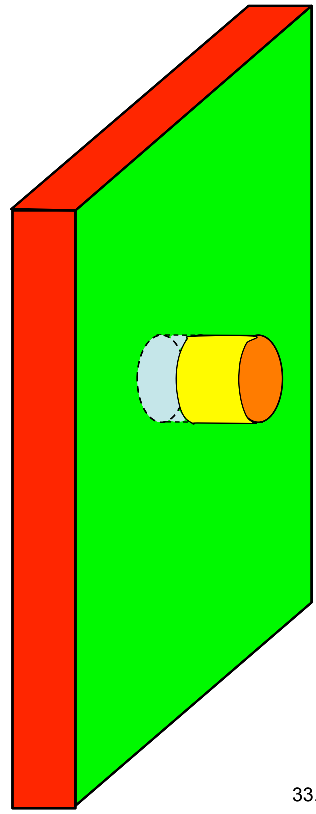
2.) The dot product of E and dA be zero (that is, the two are perpendicular to one another).

For our situation, a plain cylindrical plug will do the job.

What is shown here is a side-view and a 3-d view of the conductor, complete with Gaussian plug. Note:

a.) If the sheet extends to infinite, or if it is finite and the end-cap of the Gaussian surface lies very near both the sheet and the sheet's center, then the electric field through the end-cap will be directed outward. This is the same direction as the end-cap's area vector A. As such, all the flux will be through the right end-cap (the left end-cap is in an area where there is NO electric field) and the total electric flux through the Gaussian surface will equal EA.





The total amount of charge inside the Gaussian surface is the amount of charge on the section of conductor that is inside the Gaussian surface. That charge will equal the *charge per unit area* σ times the *area*, or A (it is the same area as that of the end-cap). With that, and noting that the flux only passing through the right-hand end-cap, Gauss's Law yields:

$$\int E \bullet dS = \frac{q_{\text{enclosee}}}{\varepsilon_o}$$

$$\Rightarrow \quad EA = \frac{\sigma A}{\varepsilon_o}$$

$$\Rightarrow \quad E_{\text{infinite, charged, conducting sheet}} = \frac{\sigma}{\varepsilon_o}$$

Note that this field is not a function of *distance from the sheet*. This was explained in class. If you don't understand it, come talk to me. (Clearly, if the sheet is not infinite, the expression will be a good approximation to some distance out, though that distance will depend upon how good an approximation you want.)

HAVING DONE THAT, now consider an infinite (or at least very large), charged, INSULATING sheet.

At least part of the problem here is that the charge-per-unit-area function σ now means something *different* than it did when used with a conductor. For an insulator, it is assumed that charge is shot throughout the structure. That means that the σ function doesn't't' tell you how much charge is on the surface, it tells you how much charge is *below* the surface (under a given area). What this means is that we can not have our Gaussian surface terminate *inside* the slab!

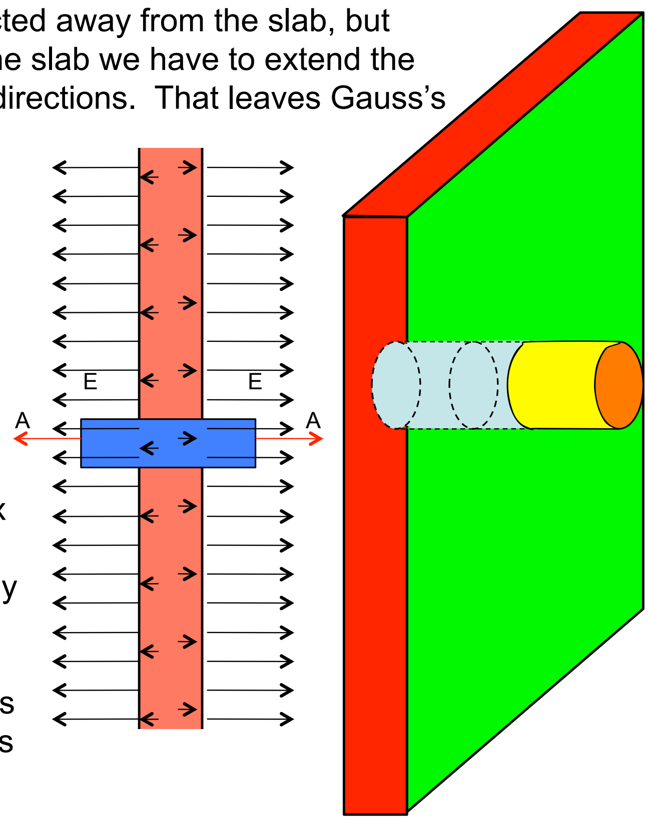
Both a side-view and a 3-d view of our slab and Gaussian plug are shown on the next page.

The electric field E is still directed away from the slab, but because it is not zero inside the slab we have to extend the Gaussian surface out in both directions. That leaves Gauss's law looking like:

$$\int \mathbf{E} \bullet d\mathbf{S} = \frac{q_{enclosee}}{\varepsilon_o}$$

$$\Rightarrow \quad 2(EA) = \frac{\sigma A}{\varepsilon_o}$$

$$\Rightarrow \quad E_{\text{inf, chrgd, insultng sht}} = \frac{\sigma}{2\varepsilon_o}$$



Notice that because it's the flux *through* the Gaussian surface end-caps that matters, we really don't care what's happening *inside* the conductor. This makes life a lot easier as what's going on inside the conductor is not a trivial matter.